ECON 897 Test (Week 4)
Aug 5, 2016

Important: This is a closed-book test. No books or lecture notes are permitted. You have **90** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. **(15 points)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by:

$$f(x, y) = (xy, x\cos y, x\sin y)$$
$$g(x, y, z) = 2x^2y + e^{yz} + zx$$
$$h(x, y, z) = \begin{cases} z \cdot \left(\frac{x^2y}{x^2+y^2}\right) & \text{if } x, y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are the functions f, g and h differentiable? Be sure to say why they are or why they are not.

Proof. f and g are differentiable because their partial derivatives exist and are <u>continuous</u>. The derivative representations are the matrices of partial derivatives. h is not differentiable, since it is not continuous, for example, at $(0, 0, 1)$. We proved this in class. □

(b) Define $s = g \circ f$. Find the representation matrix of $(Ds)_{(x,y)}$.

Proof. Apply the chain rule: $(Ds)_{(x,y)} = (Df)_{f(x,y)} \cdot (Df)_{(x,y)}$ □

(c) Do the representation matrices of $(D^2f)_{(x,y)}$ and $(D^2g)_{(x,y,z)}$ exist? If they do, find them.

Proof. The representation matrix for the second derivative of g exists, since the function goes to $\mathbb{R}$. The representation matrix is the hessian (matrix of second partials). However, there does not exist a matrix representation of $(D^2f)_{(x,y)}$, since the function is a bilinear form to $\mathbb{R}^3$. □

2. **(20 points)** A firm currently produces a good according to the production:

$$f(k,l) = Ak^{\alpha}l^{\beta}$$

The factors of production k and l are hired at market prices r and w, respectively. The firm chooses the amounts of labor and capital hired that maximize profits, which are given by the first order conditions:

$$\alpha PAk^{\alpha-1}l^{\beta} = r, \qquad \beta PAk^{\alpha}l^{\beta-1} = w$$

(a) Compute the hessian of f.

Proof.

$$\begin{bmatrix} \alpha(\alpha-1)Ak^{\alpha-2}l^{\beta} & \alpha\beta Ak^{\alpha-1}l^{\beta-1} \\ \alpha\beta Ak^{\alpha-1}l^{\beta-1} & \beta(\beta-1)Ak^{\alpha}l^{\beta-2} \end{bmatrix}$$

□

(b) Use the Implicit Function Theorem to find expressions for the derivatives of k and l with respect to the prices P, w and r? State clearly any assumptions we have to make.

Proof. Define:

$$B = P \cdot \begin{bmatrix} \alpha(\alpha-1)Ak^{\alpha-2}l^{\beta} & \alpha\beta Ak^{\alpha-1}l^{\beta-1} \\ \alpha\beta Ak^{\alpha-1}l^{\beta-1} & \beta(\beta-1)Ak^{\alpha}l^{\beta-2} \end{bmatrix}, \quad A = \begin{bmatrix} \alpha Ak^{\alpha-1}l^{\beta} & 1 & 0 \\ \beta Ak^{\alpha}l^{\beta-1} & 0 & 1 \end{bmatrix}$$

A sufficient condition for k and l to be defined as implicit functions of P, r and w is that B be an invertible matrix. This happens, for example, if $\alpha + \beta \neq 1$. Assuming this, the derivatives are given by the matrix $B^{-1} \cdot A$. □

3. **(20 points)** Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $f(x) = \frac{h(x)}{g(x)}$ and $h(x), g(x) > 0$ for all $x \in \mathbb{R}^n_+$. Assume h is concave and g is convex. Show that f is quasiconcave.

Proof. Let $x, x' \in X$, $\lambda \in [0,1]$ and $x^{\lambda} = \lambda x + (1-\lambda)x'$. By concavity of h and convexity of g:

$$f(x^{\lambda}) = \frac{h(x^{\lambda})}{g(x^{\lambda})} \geq \frac{\lambda h(x) + (1-\lambda)h(x')}{\lambda g(x) + (1-\lambda)g(x')}$$

Assume, without loss of generality, that $f(x) \geq f(x')$. Then:

$$\begin{aligned} f(x^{\lambda}) = \frac{h(x^{\lambda})}{g(x^{\lambda})} &\geq \frac{\lambda h(x) + (1-\lambda)h(x')}{\lambda g(x) + (1-\lambda)g(x')} \\ &= \frac{\lambda f(x)g(x) + (1-\lambda)f(x')g(x')}{\lambda g(x) + (1-\lambda)g(x')} \\ &\geq \frac{\lambda f(x')g(x) + (1-\lambda)f(x')g(x')}{\lambda g(x) + (1-\lambda)g(x')} \\ &= f(x') \cdot \frac{\lambda g(x) + (1-\lambda)g(x')}{\lambda g(x) + (1-\lambda)g(x')} \\ &= f(x') = \min\{f(x), f(x')\} \end{aligned}$$

Which means that f is quasiconcave. $\square$

4. **(20 points)** Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ non-empty and convex set. Assume f is a strictly quasiconcave function.

   (a) Show that any local maximum is a global maximum.

   Proof. Let $x_0 \in X$ be a local maximum. Then, there exists $\epsilon > 0$ such that $f(x_0) \geq f(x)$ for all $x \in B(x_0, \epsilon)$. Assume x_0 is not a global maximum, so there exists $x_1 \in X$ such that $f(x_1) > f(x_0)$. Since X is convex $x_\lambda = \lambda x_1 + (1-\lambda)x_0 \in X$. Choosing λ sufficiently close to 0,

   $$f(x_\lambda) \leq f(x_0) = \min\{f(x_0), f(x_1)\}$$

   which is a contradiction to f being strictly quasiconcave. $\square$

   (b) Argue that any global maximum must be unique.

   Proof. Assume x and x' are global maxima, $x \neq x'$. Given that f is strictly quasiconcave, $f(\lambda x + (1-\lambda)x') > f(x), \lambda \in (0,1)$, which contradicts x being global maximum. Thus, $x = x'$ and any global maximum is unique. $\square$

5. **(25 points)** Suppose $f : (a, b) \to \mathbb{R}$ is concave.

   (a) Fix any x_0 and define

   $$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \forall x \in (a,b)\backslash\{x_0\}.$$

   Prove that g is a decreasing function.

   Proof. Pick any $x < x'$. There are three cases $x_0 < x < x'$, $x < x_0 < x'$ and $x < x' < x_0$. We show the case $x_0 < x < x'$, other cases are similar. Because

   $$x = \frac{x' - x}{x' - x_0} x_0 + \frac{x - x_0}{x' - x_0} x',$$

   we have

   $$f(x) \geq \frac{x' - x}{x' - x_0} f(x_0) + \frac{x - x_0}{x' - x_0} f(x').$$

   But this is equivalent to

   $$\frac{f(x') - f(x_0)}{x' - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0}.$$

   $\square$

(b) Use part (a) to prove f is continuous.

Proof. Fix any $x_0 \in (a,b)$. There exists $\overline{x}, \underline{x} \in (a,b)$ such that $\underline{x} < x_0 < \overline{x}$. Then for any x between $\underline{x}$ and $\overline{x}$, we have

$$\frac{f(\overline{x}) - f(x_0)}{\overline{x} - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(\underline{x}) - f(x_0)}{\underline{x} - x_0}.$$

Assume $x > x_0$. Then the above inequalities imply

$$\frac{f(\overline{x}) - f(x_0)}{\overline{x} - x_0}(x - x_0) \leq f(x) - f(x_0) \leq \frac{f(\underline{x}) - f(x_0)}{\underline{x} - x_0}(x - x_0).$$

Sandwich theorem implies $\lim_{x \to x_0+} \big[f(x) - f(x_0)\big] = 0$. Similarly, if $x < x_0$ then the above inequalities imply

$$\frac{f(\underline{x}) - f(x_0)}{\underline{x} - x_0}(x - x_0) \leq f(x) - f(x_0) \leq \frac{f(\overline{x}) - f(x_0)}{\overline{x} - x_0}(x - x_0).$$

Again sandwich theorem implies $\lim_{x \to x_0-} \big[f(x) - f(x_0)\big] = 0$.

Remark: Note that the limits are taken when $x \longrightarrow x_0$, while leaving $\underline{x}$ and $\bar{x}$ fixed. The function might not be differentiable, so limits as $\underline{x}$ or $\bar{x}$ tend to x_0 might not exist. ☐